ECON 4130

HG 2005 H

Sketch answers to the ordinary exam 2004 H

(Note that the answers given are just sketchy. The exam paper would normally require some more commenting or details.)

Problem 1

1a.

$$\int_0^1 x^{\theta} dx = \frac{1}{\theta+1}.$$ Graph …..

1b.

$F(x) = x^5$,

(i) $P(0,5 \le X \le 0,8) = 0,8^5 - 0,5^5 = 0,328 - 0,031 = 0,297$,

(ii) $P(X > 0,8) = 1 - 0,8^5 = 1 - 0,328 = 0,672$,

(iii) Median: $m^5 = 0,5 \Rightarrow m = 0,5^{1/5} = 0,871$

1c.

$$\mathrm{E}\left(X^r\right) = \frac{\theta+1}{\theta+1+r} \Rightarrow \mathrm{E}(X) = \frac{\theta+1}{\theta+2}$$ and $$\mathrm{E}\left(X^2\right) = \frac{\theta+1}{\theta+3}$$ from which

$$\mathrm{var}(X) = \frac{\theta+1}{\theta+3} - \left(\frac{\theta+1}{\theta+2}\right)^2 = \frac{(\theta+1)\left[(\theta+2)^2 - (\theta+1)(\theta+3)\right]}{(\theta+3)(\theta+2)^2} = \frac{\theta+1}{(\theta+3)(\theta+2)^2}$$

1d.

$0 < X \le 1 \Rightarrow 0 \le Y < \infty$. For $y > 0$ we have

$P(Y \le y) = P(-\ln(X) \le y) = P(\ln(X) \ge -y) = P(X \ge e^{-y}) = e^{-y(\theta+1)}$, which is the cdf of $\exp(\theta+1)$.

Problem 2

2a.

$$(1-0,8^{\theta+1} \geq 0,75) \Leftrightarrow (0,8^{\theta+1} \leq 0,25) \Leftrightarrow ((\theta+1)\ln(0,8) \leq \ln(0,25))$$

$$\Leftrightarrow \left(\theta \geq \frac{\ln(0,25)}{\ln(0,8)} - 1 = 5,21 \right)$$

2b.

MME: $\frac{\tilde{\theta}+1}{\tilde{\theta}+2} = \bar{X}$ gives $\tilde{\theta} = \frac{2\bar{X}-1}{1-\bar{X}} = 6,4204$ where $\bar{X} = 0,88124$

MLE: Log likelihood: $l(\theta) = n\ln(\theta+1) + \theta\sum_i \ln(x_i)$ with

derivative $l'(\theta) = \frac{n}{\theta+1} + \sum_i \ln(x_i)$ giving $\hat{\theta} = -\frac{n}{\sum_i \ln(X_i)} - 1 = 6,3180$

2c.

The (weak) law of large numbers. $Y_i = \ln(X_i)$ are *iid* with constant expectation, $-1/(\theta+1)$ and variance in **1c**. The probability limit follows then by Tsjebysjev's inequality. Since $\hat{\theta} = g(\bar{Y})$ is continuous, we have

$$\text{plim}(\hat{\theta}) = g(\text{p}\lim \bar{Y}) = g(-1/(\theta+1)) = -(-(\theta+1)) - 1 = \theta \quad ,$$

2d.

The mgf of $Y_i = -\ln(X_i)$ is according to **1d** the mgf of a $\Gamma(1, \theta+1)$ distribution, i.e. $M_Y(t) = \frac{\theta+1}{\theta+1-t}$. Because of the independence, the mgf of $V = \sum Y_i$ is

$M_V(t) = \left(\frac{\theta+1}{\theta+1-t}\right)^n$ which is the mgf of $\Gamma(n, \theta+1)$.

2e.

$$\mathrm{E}\left(\frac{1}{V}\right)=\frac{\lambda^{\alpha}}{\Gamma(\alpha)}\int_0^\infty v^{\alpha-2}e^{-\lambda v}dv=\lambda\frac{\Gamma(\alpha-1)}{\Gamma(\alpha)}\frac{\lambda^{\alpha-1}}{\Gamma(\alpha-1)}\int_0^\infty v^{\alpha-2}e^{-\lambda v}dv=\frac{\lambda}{\alpha-1}$$

We find $\mathrm{E}(\hat{\theta})=\mathrm{E}\left(\frac{n}{V}-1\right)=n\frac{\theta+1}{n-1}-1=\frac{n}{n-1}\theta+\frac{1}{n-1}$ which $\to\theta$ as $n\to\infty$.